Section 4.5: Proving the Correctness of Grammars

In this section, we consider a technique for proving the correctness of grammars. We begin with a useful definition.

(4.5) Π-languages

Suppose G is a grammar and $a \in Q_G \cup \mathbf{alphabet}(G)$. Then $\Pi_{G,a} = \{\, w \in \mathbf{alphabet}(G)^* \mid$ there is a $pt \in \mathbf{PT}$ such that pt is valid for G, $\mathbf{rootLabel}(pt) = a$ and $\mathbf{yield}(pt) = w\,\}$. If it's clear which grammar we are talking about, we sometimes abbreviate $\Pi_{G,a}$ to Π_a.

For example, if G is the grammar

$$\mathsf{A} \to \mathsf{0A3}, \qquad \mathsf{A} \to \mathsf{B},$$
$$\mathsf{B} \to \mathsf{1B2}, \qquad \mathsf{B} \to \%,$$

then $\Pi_0 = \{0\}$, $\Pi_1 = \{1\}$, $\Pi_2 = \{2\}$, $\Pi_3 = \{3\}$, $\Pi_{\mathsf{A}} = \{\, 0^n 1^m 2^m 3^n \mid n, m \in \mathbb{N}\,\} = L(G)$ and $\Pi_{\mathsf{B}} = \{\, 1^m 2^m \mid m \in \mathbb{N}\,\}$.

(4.5) Π-languages (Cont.)

Proposition 4.5.1

Suppose G is a grammar.

(1) *For all $a \in \mathbf{alphabet}(G)$, $\Pi_{G,a} = \{a\}$.*

(2) *For all $q \in Q_G$, $\Pi_{G,q} = \{\, w_1 \cdots w_n \mid$ there are $a_1, \ldots, a_n \in \mathbf{Sym}$ such that $q \to a_1 \cdots a_n \in P_G$ and $w_1 \in \Pi_{G,a_1}, \ldots, w_n \in \Pi_{G,a_n} \,\}$.*

Suppose G is a grammar, and $\mathsf{A} \to \%$ and $\mathsf{A} \to \mathsf{0B1C}$ are productions of G, where $\mathsf{B}, \mathsf{C} \in Q_G$ and $0, 1 \in \mathbf{alphabet}(G)$.

By Part (2) of the proposition, in the case when $n = 0$, we have that, since $\mathsf{A} \to \% \in P_G$, then $\% \in \Pi_{\mathsf{A}}$.

Suppose $w_2 \in \Pi_{\mathsf{B}}$ and $w_4 \in \Pi_{\mathsf{C}}$. By Part (1) of the proposition, we have that $\mathsf{0} \in \Pi_{\mathsf{0}}$ and $1 \in \Pi_1$. Thus, since $\mathsf{A} \to \mathsf{0B1C} \in P_G$, we have that $0w_2 1w_4 \in \Pi_{\mathsf{A}}$.

(4.5) Techniques for Showing Grammar Correctness

If a grammar has no productions of the form $q \to r$ or $q \to \%$, for a variable r, we could use strong string induction and Proposition 4.5.1 to prove it correct.

Because this technique doesn't work in the general case, we will introduce an induction principle that will work in general.

(4.5) Principle of Induction on Π

Suppose G is a grammar and that, for all $a \in Q_G \cup \mathbf{alphabet}(G)$, $P_a(w)$ is a property of a string $w \in \Pi_{G,a}$. The *principle of induction on* Π says that

$$\text{for all } a \in Q_G \cup \mathbf{alphabet}(G), \text{ for all } w \in \Pi_{G,a},\ P_a(w)$$

follows from showing

(1) for all $a \in \mathbf{alphabet}(G)$, $P_a(a)$;

(2) for all $q \in Q_G$, $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbf{Sym}$, if $q \to a_1 \cdots a_n \in P_G$, then

for all $w_1 \in \Pi_{G,a_1}$, ..., $w_n \in \Pi_{G,a_n}$,
if (†) $P_{a_1}(w_1)$, ..., $P_{a_n}(w_n)$,
then $P_q(w_1 \cdots w_n)$.

We refer to the formula (†) as the *inductive hypothesis*.

(4.5) Principle of Induction on Π

If $a \in \mathbf{alphabet}(G)$, then $\Pi_{G,a} = \{a\}$. We will only apply the property $P_a(\cdot)$ to elements of $\Pi_{G,a}$, i.e., to a, and Part (1) requires that $P_a(a)$ holds. Thus, when applying our induction principle, we can implicitly assume that $P_a(w)$ says "$w = a$". Given this assumption, we won't have to explicitly prove Part (1).

Furthermore, when proving Part (2), given a symbol $a_i \in \mathbf{alphabet}(G)$, we will have that $w_i = a_i$, and it will be unnecessary to assume that $P_{a_i}(a_i)$, since this will always be true.

For example, given the production $A \to \mathsf{0B1C}$, where $\mathsf{B}, \mathsf{C} \in Q_G$ and $0, 1 \in \mathbf{alphabet}(G)$, we will proceed as follows. We will assume that $w_2 \in \Pi_{\mathsf{B}}$ and $w_4 \in \Pi_{\mathsf{C}}$, and that the inductive hypothesis holds: $P_{\mathsf{B}}(w_2)$ and $P_{\mathsf{C}}(w_4)$. Then, we will prove that $P_{\mathsf{A}}(0w_2 1w_4)$. Of course, we could use the variables x and y instead of w_2 and w_4.

(4.5) Example Correctness Proof

Let G be the grammar

$$\mathsf{A} \rightarrow \mathsf{0A3}, \qquad \mathsf{A} \rightarrow \mathsf{B},$$
$$\mathsf{B} \rightarrow \mathsf{1B2}, \qquad \mathsf{B} \rightarrow \%.$$

Let

$$X = \{\, \mathsf{0}^n \mathsf{1}^m \mathsf{2}^m \mathsf{3}^n \mid n, m \in \mathbb{N} \,\},$$
$$Y = \{\, \mathsf{1}^m \mathsf{2}^m \mid m \in \mathbb{N} \,\}.$$

To prove that $L(G) = X$, it will suffice to show that $X \subseteq L(G) \subseteq X$.

(4.5) $X \subseteq L(G)$

First, we prove that $Y \subseteq \Pi_{\mathsf{B}}$. It will suffice to show that, for all $m \in \mathbb{N}$, $\mathsf{1}^m \mathsf{2}^m \in \Pi_{\mathsf{B}}$. We proceed by mathematical induction.

(Basis Step) Because $\mathsf{B} \rightarrow \% \in P$, we have that $\mathsf{1}^0 \mathsf{2}^0 = \%\% = \% \in \Pi_{\mathsf{B}}$, by Proposition 4.5.1.

(Inductive Step) Suppose $m \in \mathbb{N}$, and assume the inductive hypothesis: $\mathsf{1}^m \mathsf{2}^m \in \Pi_{\mathsf{B}}$. Because $\mathsf{B} \rightarrow \mathsf{1B2} \in P$, it follows that $\mathsf{1}^{m+1} \mathsf{2}^{m+1} = \mathsf{1}(\mathsf{1}^m \mathsf{2}^m)\mathsf{2} \in \Pi_{\mathsf{B}}$, by Proposition 4.5.1.

(4.5) $X \subseteq L(G)$ (Cont.)

Next, we prove that $X \subseteq \Pi_{\mathsf{A}} = L(G)$. It will suffice to show that, for all $n, m \in \mathbb{N}$, $0^n1^m2^m3^n \in \Pi_{\mathsf{A}}$. Suppose $m \in \mathbb{N}$. It will suffice to show that, for all $n \in \mathbb{N}$, $0^n1^m2^m3^n \in \Pi_{\mathsf{A}}$. We proceed by mathematical induction.

(Basis Step) Because $Y \subseteq \Pi_{\mathsf{B}}$, we have that $1^m2^m \in \Pi_{\mathsf{B}}$. Then, since $\mathsf{A} \rightarrow \mathsf{B} \in P$, we have that $0^01^m2^m3^0 = \%1^m2^m\% = 1^m2^m \in \Pi_{\mathsf{A}}$, by Proposition 4.5.1.

(Inductive Step) Suppose $n \in \mathbb{N}$, and assume the inductive hypothesis: $0^n1^m2^m3^n \in \Pi_{\mathsf{A}}$. Because $\mathsf{A} \rightarrow 0\mathsf{A}3 \in P$, it follows that $0^{n+1}1^m2^m3^{n+1} = 0(0^n1^m2^m3^n)3 \in \Pi_{\mathsf{A}}$, by Proposition 4.5.1.

(4.5) $L(G) \subseteq X$

Since $\Pi_{\mathsf{A}} = L(G)$, it will suffice to show that

(A) for all $w \in \Pi_{\mathsf{A}}$, $w \in X$;

(B) for all $w \in \Pi_{\mathsf{B}}$, $w \in Y$.

We proceed by induction on Π.

Formally, this means that we let the properties $P_{\mathsf{A}}(w)$ and $P_{\mathsf{B}}(w)$ be "$w \in X$" and "$w \in Y$", respectively, and then use the induction principle to prove that, for all $a \in Q_G \cup \mathbf{alphabet}(G)$, for all $w \in \Pi_a$, $P_a(w)$. But we will actually work more informally.

There are four productions to consider.

- ($\mathsf{A} \rightarrow 0\mathsf{A}3$) Suppose $w \in \Pi_{\mathsf{A}}$, and assume the inductive hypothesis: $w \in X$. We must show that $0w3 \in X$. Because $w \in X$, we have that $w = 0^n1^m2^m3^n$, for some $n, m \in \mathbb{N}$. Thus $0w3 = 0(0^n1^m2^m3^n)3 = 0^{n+1}1^m2^m3^{n+1} \in X$.

(4.5) $L(G) \subseteq X$ (Cont.)

- (A $\rightarrow$ B) Suppose $w \in \Pi_{\mathsf{B}}$, and assume the inductive hypothesis: $w \in Y$. We must show that $w \in X$. Because $w \in Y$, we have that $w = 1^m 2^m$, for some $m \in \mathbb{N}$. Thus $w = \% w \% = 0^0 1^m 2^m 3^0 \in X$.
- (B $\rightarrow$ 1B2) Suppose $w \in \Pi_{\mathsf{B}}$, and assume the inductive hypothesis: $w \in Y$. We must show that $1w2 \in Y$. Because $w \in Y$, we have that $w = 1^m 2^m$, for some $m \in \mathbb{N}$. Thus $1w2 = 1(1^m 2^m)2 = 1^{m+1} 2^{m+1} \in Y$.
- (B $\rightarrow$ %) We must show that $\% \in Y$, and this follows since $\% = \%\% = 1^0 2^0 \in Y$.

(4.5) Correctness Proof (Cont.)

If we look at the proofs of $X \subseteq L(G)$ and $L(G) \subseteq X$, we can conclude that, for all $w \in \mathbf{Str}$:

(A) $w \in \Pi_{\mathsf{A}}$ iff $w \in X$; and

(B) $w \in \Pi_{\mathsf{B}}$ iff $w \in Y$.