

Section 2.2: String Induction Principles

In this section, we introduce three string induction principles: left string induction, right string induction and strong string induction. These induction principles are ways of showing that every string $w \in A^*$ has property $P(w)$, where A is some set of symbols.

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The \LaTeX source of these slides, the associated book, and the distribution of the Forlan toolset are available on the WWW at <http://people.cis.ksu.edu/~stough/forlan/>.

(2.2) The Principle of Left String Induction

Suppose $A \subseteq \mathbf{Sym}$. The *principle of left string induction* for A says that

for all $w \in A^*$, $P(w)$

follows from showing

- (basis step)

$P(\epsilon)$;

- (inductive step)

for all $a \in A$ and $w \in A^*$, if $P(w)$ then $P(aw)$

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We refer to the formula (\dagger) as the *inductive hypothesis*.

(2.2) The Principle of Right String Induction

Suppose $A \subseteq \mathbf{Sym}$. The *principle of right string induction* for A says that

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(2.2) The Principle of Strong String Induction

Suppose $A \subseteq \mathbf{Sym}$. The *principle of strong string induction* for A says that

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Suppose $A \subseteq \mathbf{Sym}$. The *principle of strong string induction* for A says that

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if (\dagger) for all $x \in A^*$, if $|x| < |w|$, then $P(x)$,
then $P(w)$.

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(2.2) Example Proof Using Strong String Induction

Let X be the least subset of $\{0, 1\}^*$ such that:

- (1) $\% \in X$;
- (2) for all $a \in \{0, 1\}$ and $x \in X$, $axa \in X$.

Thus, X contains the elements:

$\%$,

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We will show that $X = Y$, where $Y = \{w \in \{0, 1\}^* \mid w \text{ is a palindrome and } |w| \text{ is even}\}$.

(2.2) Proof that $Y \subseteq X$

Lemma 2.2.4

$Y \subseteq X$.

Proof. Since $Y \subseteq \{0, 1\}^*$, it will suffice to show that, for all $w \in \{0, 1\}^*$,

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We proceed by strong string induction.

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Proof (cont.). Suppose $w \in Y$, so that w is a palindrome and $|w|$ is even. It remains to show that $w \in X$.

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Proof (cont.). Suppose $w \in Y$, so that w is a palindrome and $|w|$ is even. It remains to show that $w \in X$. If $w = \%$, then $w = \% \in X$, by Part (1) of the definition of X . So, suppose $w \neq \%$.

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(2.2) Proof that $Y \subseteq X$ (Cont.)

Proof (cont.). Suppose $w \in Y$, so that w is a palindrome and $|w|$ is even. It remains to show that $w \in X$. If $w = \epsilon$, then $w = \epsilon \in X$, by Part (1) of the definition of X . So, suppose $w \neq \epsilon$. Since $|w| \geq 2$, we have that $w = axb$ for some $a, b \in \{0, 1\}$ and $x \in \{0, 1\}^*$. And, $|x|$ is even. Furthermore, because w is a palindrome, it follows that $a = b$ and x is a palindrome. Thus $w = axa$ and $x \in Y$.

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(2.2) Proof that $Y \subseteq X$ (Cont.)

Proof (cont.). Suppose $w \in Y$, so that w is a palindrome and $|w|$ is even. It remains to show that $w \in X$. If $w = \epsilon$, then $w = \epsilon \in X$, by Part (1) of the definition of X . So, suppose $w \neq \epsilon$. Since $|w| \geq 2$, we have that $w = axb$ for some $a, b \in \{0, 1\}$ and $x \in \{0, 1\}^*$. And, $|x|$ is even. Furthermore, because w is a palindrome, it follows that $a = b$ and x is a palindrome. Thus $w = axa$ and $x \in Y$. Since $|x| < |w|$, the inductive hypothesis tells us that

if $x \in Y$, then $x \in X$.

But $x \in Y$, and thus $x \in X$.

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(2.2) Proof that $Y \subseteq X$ (Cont.)

Proof (cont.). Suppose $w \in Y$, so that w is a palindrome and $|w|$ is even. It remains to show that $w \in X$. If $w = \epsilon$, then $w = \epsilon \in X$, by Part (1) of the definition of X . So, suppose $w \neq \epsilon$. Since $|w| \geq 2$, we have that $w = axb$ for some $a, b \in \{0, 1\}$ and $x \in \{0, 1\}^*$. And, $|x|$ is even. Furthermore, because w is a palindrome, it follows that $a = b$ and x is a palindrome. Thus $w = axa$ and $x \in Y$. Since $|x| < |w|$, the inductive hypothesis tells us that

if $x \in Y$, then $x \in X$.

But $x \in Y$, and thus $x \in X$. Thus, by Part (2) of the definition of X , we have that $w = axa \in X$. \square

(2.2) Proof that $X \subseteq Y$

Lemma 2.2.5

$X \subseteq Y$.

We could prove this lemma by strong string induction. But it is simpler and more elegant to use an alternative approach, using the induction principle that comes from the inductive definition of X .

(2.2) Principle of Induction on X

The *principle of induction on X* says that

for all $w \in X$, $P(w)$

follows from showing

(1)

$P(\%)$;

(2)

for all $a \in \{0, 1\}$ and $x \in X$, if (\dagger) , then $P(axa)$.

We refer to the formula (\dagger) as the *inductive hypothesis*.

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for all $a \in \{0, 1\}$ and $x \in X$, if $(\dagger) P(x)$, then $P(axa)$.

We refer to the formula (\dagger) as the *inductive hypothesis*.

(2.2) Proof that $X \subseteq Y$ (Cont.)

Proof. We use induction on X to show that, for all $w \in X$, $w \in Y$.

There are two steps to show.

- (1) Since $\%_0$ is a palindrome and $|\%_0| = 0$ is even, we have that $\%_0 \in Y$.
- (2) Let $a \in \{0, 1\}$ and $x \in X$. Assume the inductive hypothesis: $x \in Y$. Since x is a palindrome, we have that axa is also a palindrome. And, because $|axa| = |x| + 2$ and $|x|$ is even, it follows that $|axa|$ is even. Thus $axa \in Y$, as required.

□

(2.2) Proof that $X = Y$

Proposition 2.2.6

$$X = Y.$$

Proof. Follows immediately from Lemmas 2.2.4 and 2.2.5. \square