Section 1.2: Induction Principles for the Natural Numbers

In this section, we consider two methods for proving that every natural number n has some property $P(n)$. The first method is the familiar principle of mathematical induction. The second method is the principle of strong (or course-of-values) induction.

(1.2) The Principle of Mathematical Induction

The *principle of mathematical induction* says that

$$\text{for all } n \in \mathbb{N},\ P(n)$$

follows from showing

- (basis step)

$$P(0);$$

- (inductive step)

$$\text{for all } n \in \mathbb{N},\ \text{if } (\dagger)\ P(n), \text{ then } P(n+1).$$

We refer to the formula (†) as the *inductive hypothesis*.

(1.2) The Principle of Strong Induction

The *principle of strong induction* says that

for all $n \in \mathbb{N}$, $P(n)$

follows from showing

for all $n \in \mathbb{N}$,
if (‡) for all $m \in \mathbb{N}$, if $m < n$, then $P(m)$,
then $P(n)$.

We refer to the formula (‡) as the *inductive hypothesis*.

(1.2) Example Proof Using Strong Induction

Proposition 1.2.2
Every nonempty set of natural numbers has a least element.

Proof. Let X be a nonempty set of natural numbers.

We begin by using strong induction to show that, for all $n \in \mathbb{N}$,

if $n \in X$, then X has a least element.

Suppose $n \in \mathbb{N}$, and assume the inductive hypothesis: for all $m \in \mathbb{N}$, if $m < n$, then

if $m \in X$, then X has a least element.

We must show that

if $n \in X$, then X has a least element.

(1.2) Example Proof (Cont.)

Proof (cont.). Suppose $n \in X$. It remains to show that X has a least element. If n is less-than-or-equal-to every element of X, then we are done. Otherwise, there is an $m \in X$ such that $m < n$. By the inductive hypothesis, we have that

if $m \in X$, then X has a least element.

But $m \in X$, and thus X has a least element. This completes our strong induction.

(1.2) Example Proof (Cont.)

Proof (cont.). Now we use the result of our strong induction to prove that X has a least element. Since X is a nonempty subset of $\mathbb{N}$, there is an $n \in \mathbb{N}$ such that $n \in X$. By the result of our induction, we can conclude that

if $n \in X$, then X has a least element.

But $n \in X$, and thus X has a least element. $\square$

(1.2) Comparison of Induction Principles

It is easy to see that any proof using mathematical induction can be turned into one using strong induction. (Split into the cases where $n = 0$ and $n = m + 1$, for some m.)

Are there results that can be proven using strong induction but not using mathematical induction? The answer turns out to be "no". In fact, a proof using strong induction can be mechanically turned into one using mathematical induction, but at the cost of making the property $P(n)$ more complicated. Challenge: find a $P(n)$ that can be used to prove Proposition 1.2.2 using mathematical induction.

As a matter of style, one should use mathematical induction whenever it is *convenient* to do so, since it is the more straightforward of the two principles.